The Bootstrapped Multitaper F-test

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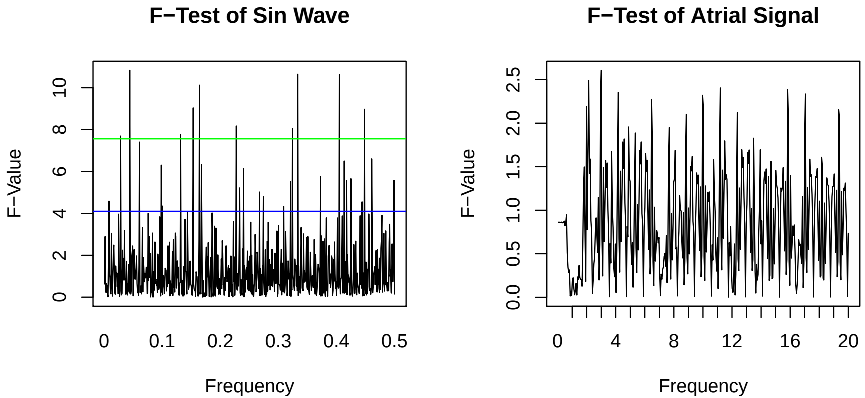
Motivation

What is an F-test, what is wrong with it and how can we improve it?



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Preliminaries - The Multitaper Method(MTM)

- It is the primary tool for spectral estimation that balances the variance and bias of the estimated spectrum.

$$\overline{S}(f) = \frac{1}{K}\sum_{k=0}^{K-1} |Y_k(f)|^2, \tag{1}$$

$$Y_k(f) = \sum_{t=0}^{N-1} \upsilon_t^{(k)} e^{-2i\pi ft} x_t, \tag{2}$$

where Y_k are the eigenspectra and $\upsilon_t^{(k)}$ are the Slepian sequences in the time domain.

- The Slepian sequences are defined for a choice of NW, with the parameters of NW and K being user selected.

Preliminaries - F-test for line components

- Using the eigenspectra that are found when performing the MTM, we can think of the F-test as essentially complex valued linear regression,

$$Y_k(f) = \mu(f) V_k(0) + \epsilon(f), \tag{3}$$

$$V_k(f) = \sum_{t=0}^{N-1} \upsilon_t^{(k)} e^{-2i\pi f t}, \tag{4}$$

where we assume $\epsilon(f) \sim CN(0, \sigma^2_{noise})$.

- After performing this regression, we are interested in testing the null hypothesis, $H_0 : \mu(f) = 0$.

Preliminaries - F-statistic calculation

- To test H_0, we calculate the F-statistic which is defined to be:

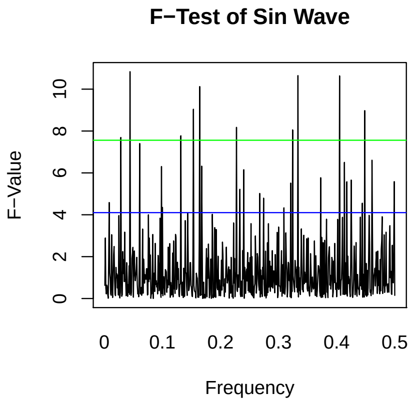
$$F(f) = (K-1)\frac{|\hat{\mu}(f)|^2 \sum_{k=0}^{K-1} |V_k(0)|^2}{\sum_{k=0}^{K-1} |\hat{r}_k(f)|^2}, \quad (5)$$

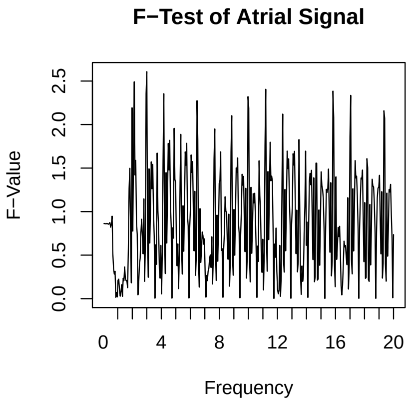
$$\hat{r}_k(f) = Y_k(f) - \hat{\mu}(f) V_k(0), \quad (6)$$

$$\hat{\mu}(f) = \frac{\sum_{k=0}^{K-1} V_k(0) Y_k(f)}{\sum_{k=0}^{K-1} |V_k(0)|^2}. \quad (7)$$

- The F-statistic should follow an $F(2, 2K-2, \alpha)$ distribution if H_0 is true.

Preliminaries - Examples





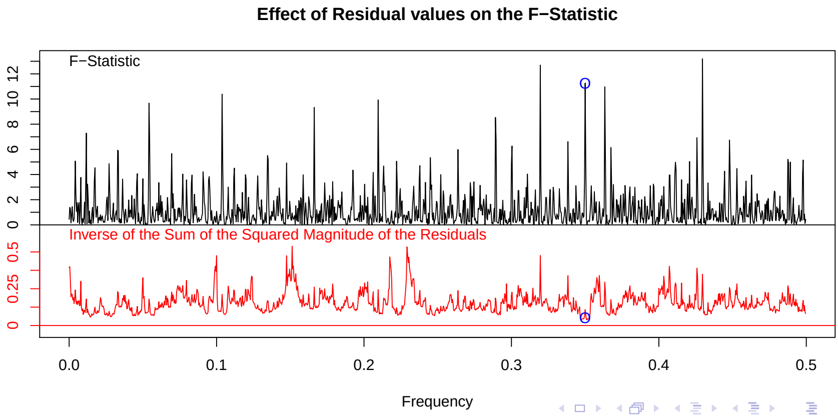
Problems With The F-Test

The test suffers lowered detection rates and higher rates of false detection when:

- There is a low signal to noise ratio.
- The choice of parameters NW and K are made incorrectly.
- The data is non-stationary.

Residual effect on *F*-Test

- After examining the residuals from the *F*-test, we noticed that the values were close to zero for some falsely detected signals and the values were quite large for actual signals.



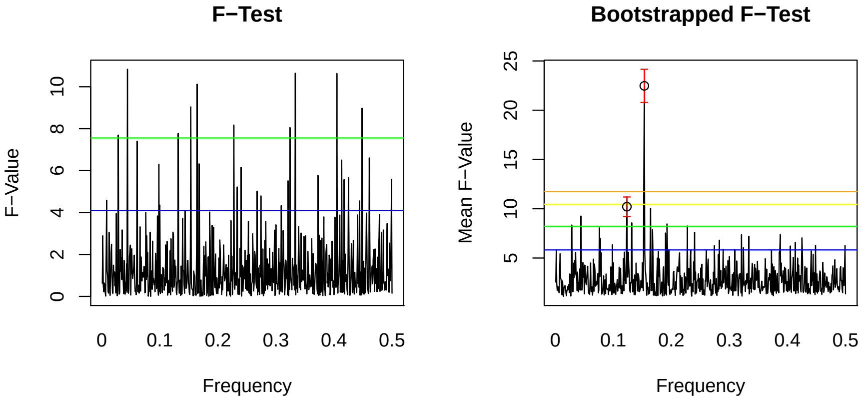
Bootstrapping residuals F-test - motivation

- Ideally, if all the structure in the residuals were removed (we have chosen NW and K correctly), they should be random variables that are independent of frequency.
- Treating them as independent realizations, we can perform a bootstrap to get a better estimate of the distribution of our F-statistic.
- We do so by re-sampling the residuals with replacement and computing a new F-value. We then take the mean of a number of re-sampled F-values to get an unbiased estimate of the location of our test statistic.

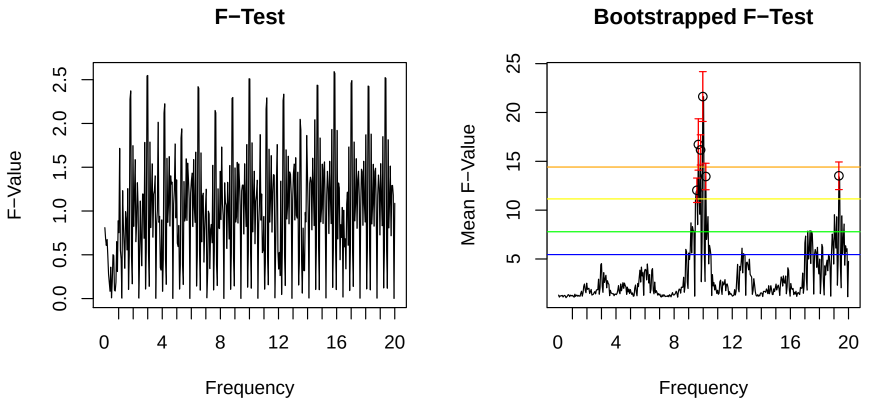
Bootstrapped F-Test Algorithm

- The Bootstrapped residuals F-test algorithm is as follows,
    1. Compute the F-test.
    2. Re-sample the residuals.
    3. Compute new values for $\hat{Y}_k^{(1)}(f)$, $\hat{\mu}^{(1)}(f)$, $\hat{r}_k^{(1)\prime}(f)$, and $\hat{F}^{(1)}(f)$.
    4. Repeat steps 2 and 3 M times ($M > 100$) and take the mean of F-statistics found for each re-sampling.
    5. Lastly to test for signal detection we check if this value exceeds an empirical cut-off to determined by the noise level of the time series and parameter choices..

Bootstrapped *F*-Test Example Sin Wave



Bootstrapped *F*-Test Example Atrial ECG

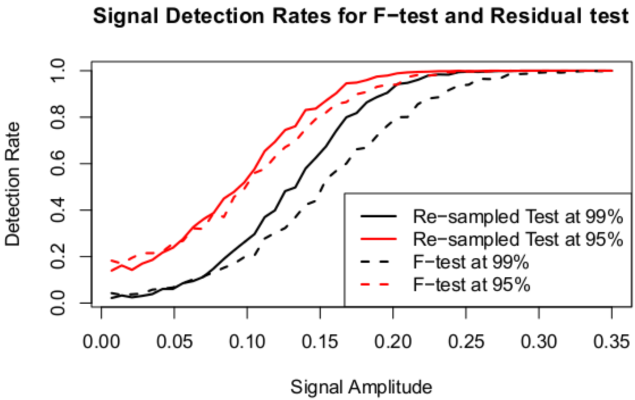


Bootstrapped F-Test - Simulations

- We wanted to compare the ability of the Bootstrapped F-test to identify a single sinusoid with varying levels of amplitude within Gaussian noise of constant variance to that of the F-test.
- We simulated a data set, $Y_t = \alpha \sin(2\pi(.125t)) + z_t$, with $z_t \sim N(0,1)$, the amplitude of the signal, α varied across $(0, .5]$.
- We used $N = 1000$, $NW = 4$, and $K = 7$ for both tests.

Test comparison results

- The result of the tests was that the re-sampled F-test had higher detection rates for signals of low to moderate power and performed equally well otherwise.



Acknowledgments

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